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Systems of Pencils of Lines in Ordinary Space.

BY ALTON L. MILLER.

In his classic paper entitled "Preliminari di una Teoria delle Varietá Luoghi di Spazi," Segre[*] laid the foundation of investigations of the projective differential properties of geometric configurations in n-dimensions by synthetic methods. It is the purpose of this paper to apply some of the results of these investigations to the study of families of pencils of lines in ordinary space. By means of the Klein coordinates of a line, there is set up a one to one correspondence between lines of space and points of a hyperquadric, Q, in five dimensions. To a line on this hyperquadric there corresponds in S_3 a pencil of lines. Thus, to ruled varieties on Q there correspond families of pencils in S_3. This correspondence is discussed completely in the following paper.

Let $(x) = (x_0, x_1, x_2, x_3, x_4, x_5)$ be the Klein coordinates of a line in ordinary space, then

$$(xx) = \sum_0^5 x_i^2 = 0. \tag{1}$$

If at the same time we think of (x) as a point in five dimensions, equation (1) represents a hyperquadric, Q, in that space. Thus, to every line in ordinary space there corresponds in five dimensions a point on the hyperquadric, Q.

If two lines intersect, their coordinates satisfy the relation $(yz) = \sum_0^5 z_i y_i = 0$, and the coordinates of any line in their pencil are given by $(x) = \lambda(y) + \mu(z)$ for some value of $\lambda : \mu$. The points of Q which correspond to two intersecting lines are therefore conjugate with respect to Q and every point $(x) = \lambda(y) + \mu(z)$ lies on Q, on the line from (y) to (z).

PART I.

1. A ruled surface, R, on Q corresponds in S_3 to a one-parameter family of pencils of lines, **R**. The centers of these pencils, in general, trace a curve C, while their planes, in general, envelop a developable, c. If R is developable,

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[*] *Rendiconti del Circolo Matematico di Palermo,* Vol. XXX (1910).

C is the edge of regression of c. Points on the edge of regression of R correspond to the characteristics of c. If R lies in an S_4 the corresponding $\mathbf{R}$ lies in a linear complex, and if R lies in an S_3 $\mathbf{R}$ belongs to a linear congruence. This congruence will have distinct, coplanar, or coincident directrices according as R lies in an S_3 which cuts Q in a non-degenerate quadric, a pair of planes, or a cone.

PART II.

2. A two-parameter family of lines on Q generates a ruled V_3 which we will call M. To M there corresponds in S_3 a two-parameter family of pencils generating a line complex which we will call an m-complex. Every linear complex is an m-complex. In general the centers of the ∞^2 pencils trace a surface, S, and their planes envelop a non-developable surface, s. S may reduce to a curve or even a point, and, in the dual case, s may reduce to a developable or even a plane. In general S and s do not coincide.

3. We can represent M analytically as follows: Let $x_i = \alpha_i(u, v)$ and $x_i = \beta_i(u, v)$ $i = 0, 1, \ldots, 5$ be any two surfaces on Q. That is $(\alpha\alpha) = (\beta\beta) = 0$ for all values of u, v. Furthermore, let us assume that if A and B are two corresponding points of α and β, that is two points obtained by giving u and v the same values in α and β, then the line from A to B lies entirely on Q. The necessary and sufficient condition for this is

$$(\alpha\beta) = 0 \text{ for all values of } u \text{ and } v. \tag{2}$$

Then lines joining corresponding points of α and β generate a ruled V_3 of Q, M. Hence M is given by

$$x_i = \alpha_i(u, v) + t\beta_i(u, v) \qquad i = 0, 1, \ldots, 5. \tag{3}$$

If we fix u and v in (3) but cause t to vary (x) traces a line generator of M.

In S_3 the α and β surfaces represent line congruences of such a nature that corresponding lines of each intersect. If we fix u and v in (3), but cause t to vary, (x) traces a pencil of lines, one of the generating pencils of the m-complex represented by (3).

The tangent S_3 to M at (x_0), *i. e.*, $u = u_0$, $v = v_0$, $t = t_0$, is determined by (x), (x_u), (x_v), (x_t), that is by the four points

$$(\alpha), \quad (\beta), \quad (\alpha_u + t\beta_u), \quad (\alpha_v + t\beta_v). \tag{4}$$

Let G represent the generator from α to β. If we allow t to vary in (4), but hold u and v fast, (4) determines the various tangent S_3's to M at points of G.

We will call the linear congruence corresponding to the tangent S_3 to M, *tangent* to the m-complex at the line corresponding to the point of contact of the S_3. Such a tangent linear congruence will always be degenerate in that its directrices will be coincident. It will be determined by a line of m and three lines of m infinitely near the first, but not lying in the same regulus with the first. For example by (x), $(x)+(x_u)\Delta u$, $(x)+(x_v)\Delta v$, $(x)+(x_t)\Delta t$, where Δu, Δv, Δt are all infinitesimal. Evidently the S_3 in S_5 determined by these four points is precisely the same as the S_3 determined by the points of (4), although those points do not lie on Q and therefore do not correspond to lines in ordinary space. All of the linear congruences tangent to m at lines of the pencil g have g in common, but in general, nothing else.

4. A line of M infinitely near G will join a point of the α-surface infinitely near (α) to a point of the β-surface infinitely near (β). That is a line of M infinitely near G will be determined by $(\alpha+\alpha_u\Delta u+\alpha_v\Delta v)$, $(\beta+\beta_u\Delta u+\beta_v\Delta v)$, where Δu and Δv are infinitesimal. As we vary the ratio of $\Delta u:\Delta v$ we get ∞^1 lines of M infinitely near G. G and a line of M infinitely near G determine the S_3 of the points

$$(\alpha),\quad (\beta),\quad (\alpha_u\Delta u+\alpha_v\Delta v),\quad (\beta_u\Delta u+\beta_v\Delta v). \tag{5}$$

Segre* has shown that the locus of the ∞^1 S_3's obtained by varying the ratio $\Delta u:\Delta v$ in (5) is a quadratic cone in S_5. This quadratic cone has G for a line of double points, and one other double point in each plane of Q that contains G exists for the quartic of intersection of Q with this cone. Hence the intersection of Q with the locus of all S_3's determined by G and the lines of M infinitely near G, is a quartic with double points along G and one in each of the planes of Q that contain G.

The above theorem stated in terms of line geometry becomes:

The locus of the ∞^1 linear congruences determined by a pencil g of an m-complex, and the pencils of the m-complex infinitely near g is a quadratic complex with double lines, the lines of g and one other in the plane that contains g and one other through the center of g. This is a tetrahedral complex of the type [(22) (11)].†

5. If two of the tangent S_3's at points of a generator G have a plane in common, all the S_3's tangent to M at points of G lie in an S_4. For the S_3 tangent to M at P_1 of G is the S_3 of (α), (β), $(\alpha_u+t_1\beta_u)$, $(\alpha_v+t_1\beta_v)$, and that at P_2 of (α), (β), $(\alpha_u+t_2\beta_u)$, $(\alpha_v+t_2\beta_v)$. Since these two have a plane in common, by Grassman's theorem, they lie in an S_4, and any linear combina-

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* Segre, *loc. cit.*, No. 12.

† A. Weiler, *Math. Annalen*, Vol. VII (1874). Sturm, "Liniengeometrie," III, p. 436.

tion of the above eight points lies in the same S_4. In particular the following points lie in an S_4:

$$(\alpha),\quad (\beta),\quad (\alpha_u),\quad (\beta_u),\quad (\alpha_v),\quad (\beta_v). \tag{6}$$

Hence every tangent S_3 to M at a point of G lies in this S_4. Note that it does not follow that all tangent S_3's to M at points of G have a plane in common. This is not in general true.

In line space the above theorem states: *If two of the tangent linear congruences to m at lines of a pencil g have in common besides g a second pencil h having a line in common with g, all the tangent linear congruences to m at lines of g lie in a linear complex.*

The S_3's tangent to M along a generator G all have a plane in common if, and only if, the S_3's determined by G and the lines of M infinitely near G have a plane in common. In fact the tangent S_3's of (4) are S_3's determined by G and lines of the regulus obtained when we vary t in the line from $(\alpha_u+t\beta_u)$ to $(\alpha_v+t\beta_v)$. If a plane is to be common to all these S_3's it must cut every line of this regulus. Hence the regulus cuts that plane either in a conic, which is impossible, for then G would cut the regulus twice, and all the tangents S_3's would coincide, or else in a line. Then G cuts the regulus in one point. Hence a necessary and sufficient condition that all the tangent S_3's to M along G have a plane in common is that G cut the regulus mentioned above.

But the S_3's determined by G and the lines infinitely near G are the same S_3's as those determined by G, and the lines of the regulus obtained by varying the ratio $\Delta u:\Delta v$ in the line from $(\alpha_u\Delta u+\alpha_v\Delta v)$ to $(\beta_u\Delta u+\beta_v\Delta v)$. And a necessary and sufficient condition that all these S_3's have a plane in common is that G cut this regulus. But the two reguli that determine the tangent S_3's with G and the S_3's of G and the near by lines, are conjugate reguli of the same quadric. Thus, if G cuts one regulus it cuts the other also. Hence the theorem.

In line space this theorem becomes: *The tangent linear congruences to m at lines of a pencil g have in common a second pencil h having a line in common with g if, and only if, the linear congruences determined by g and the pencils of m infinitely near g have in common a second pencil h′ having a line in common with g.*

If all the S_3's determined by G, and the lines of M infinitely near G have in common a plane of Q, they all coincide. Thus, *if all the linear congruences determined by g and the pencils of m infinitely near g have in common anything besides a second pencil having a line in common with g, they coincide.*

If all the tangent S_3's to M along the generators of M have a plane in common, M is said to be developable of the first kind. We will also call developable of the first kind the corresponding m-complexes, viz., those whose tangent linear congruences along the pencils of m have in common besides g, a second pencil having a line in common with g.

6. The linear congruence determined by two pencils is that congruence which contains the two pencils, and therefore the congruence which has for directrices the line joining the centers of the two pencils and the line of intersection of the planes of the two pencils. Hence the directrices of a linear congruence determined by a pencil g and a second pencil of m infinitely near g are a tangent line to S at the center of g, and a tangent line to s at the point of contact of g.

Let us consider the nature of these surfaces S and s when m is developable of the first kind. That is all the linear congruences whose directrices are corresponding tangent lines to S and s have in common g and a second pencil h which has a line y in common with g. Let P be the center and π the plane of g. P lies on S and π is tangent to s at P'. The tangent plane to S at P is π'. Then the directrices of the linear congruences referred to above will be corresponding lines of the pencils $P\pi'$ and $\pi P'$. If all of these congruences are to contain h, every line of h must cut every line of the pencils $P\pi'$ and $\pi P'$. Since h is different from g it must then be the pencil $P'\pi'$; and y, the line joining corresponding points of S and s, is tangent to both these surfaces. Hence the lines y form a congruence whose focal surfaces are S and s.

Conversely, the two-parameter family of pencils having as centers one set of focal points of a line congruence and planes, the corresponding focal planes of the congruence is an m-complex of the first developable kind. Thus every line congruence generates two m-complexes of this kind.

7. Segre[*] has shown that if M is a V_3 in S_5 of the first developable kind the lines of M are all tangent to a surface or else all cut a curve. Let us consider first the case in which all the lines of M are tangent to a surface. This surface lies on Q and may be taken as the α-surface. At every point of α there are two lines tangent to α and lying on Q. β may be chosen as any surface cutting one set of these lines, for example,

$$\beta = \alpha_u + \rho\alpha_v . \tag{7}$$

But β must be a point on Q, hence

$$\rho^2(\alpha_u\alpha_u) + 2\rho(\alpha_u\alpha_v) + (\alpha_v\alpha_v) = 0. \tag{8}$$

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[*] Segre, *loc. cit.*, No. 29.

The two values of ρ obtained by solving (8) are the two directions of lines tangent to α and lying on Q. Call them ρ_1 and ρ_2. Let G_1 and G_2 be the two corresponding lines tangent to α. The locus of lines G_1 will be M_1 and the locus of G_2 will be M_2. The points of (4) determining the tangent S_3 become

$$(\alpha),\quad (\alpha_u+\rho\alpha_v),\quad [\alpha_{uu}+2\rho\alpha_{uv}+\rho^2\alpha_{vv}+\alpha_v(\rho_u+\rho\rho_v)\alpha_u+t(\alpha_{uu}+\rho\alpha_{uv})+t\rho_u\alpha_v], \tag{9}$$

and the points of (5) determining the S_3 of G and a line of M infinitely near G become

$$(\alpha),\quad (\alpha_u),\quad (\alpha_v),\quad [(\alpha_{uu}+\rho\alpha_{uv})\Delta u+(\alpha_{uv}+\rho\alpha_{vv})\Delta v]. \tag{10}$$

Thus, when M is a developable of the first sort the tangent S_3*'s* to M along G, and the S_3*'s* of G and the lines of M infinitely near G form two pencils of S_3*'s* which lie in the same S_4 of (6). The plane axis of pencil (9) (as we vary t) depends on ρ, but that of (10) does not. Thus (9) furnishes two distinct pencils (9_1) and (9_2), while (10) furnishes two pencils (10_1) and (10_2) with the same central plane.

G_1 lies in two planes on Q, π_1 and π_1' and, similarly, G_2 in π_2 and π_2'. Since π_1 and π_2' are in different systems of planes of Q they intersect in a line L, and π_2 and π_1' intersect in L'. The four pencils of (9) and (10) cut on L and L' projective ranges. Projective ranges are also set up on G_1 and G_2 by means of (9_2) and (9_1), respectively.

In S_3 the congruence which corresponds to the tangent α-surface of S_5 is the y-congruence. To M_1 and M_2 there correspond the two m-complexes m_1 and m_2 mentioned in No. 6 as generated by the y-congruence. Thus, *if the m-complex is developable of the first sort and the y-congruence is not degenerate (i. e., a ruled surface) the tangent linear congruences to m along g form a pencil of congruences, and the linear congruences whose directrices are the corresponding tangent lines to the focal surfaces of the y-congruence also form a pencil of congruences. The centers (two pencils with a common line) of the last-named pencils of congruences for* m_1 *and* m_2 *coincide, while those of the first named do not.*

The lines L and L' correspond in S_3 to the pencils of tangent lines at corresponding points of S and s. Hence the above pencils of congruences cut off projective pencils of lines on these pencils.*

The necessary and sufficient condition that g and h coincide, *i. e.*, that S and s coincide is that the roots of (8) be equal. Hence

$$(\alpha_u\alpha_v)^2-(\alpha_u\alpha_u)(\alpha_v\alpha_v)=0\,† \tag{11}$$

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* These are the projectivities of Waelsch, see "Zur Infinitesimalgeometrie der Strahlencongruenzen und Flächen," *Sitzungsbericht Akad. Wien Mathem. Classe*, i. 100 Abth. IIa (1891).

† Waelsch, *loc. cit.*, obtains the same result in a different way.

is a necessary and sufficient condition that S and s coincide. It will appear later that this is a necessary and sufficient condition that M have a fixed tangent S_3 along every generator.

Since pencils (10_1) and (10_2) have the same center, a necessary and sufficient condition that they cut off the same projectivities on L and L' is that they lie in the same S_4. A necessary and sufficient condition for this is

$$|\alpha, \quad \alpha_u, \quad \alpha_v, \quad \alpha_{uu}, \quad \alpha_{uv}, \quad \alpha_{vv}| = 0, \tag{12}$$

that is, the α-surface is a ϕ-surface of Segre.*

In line space this reads: A necessary and sufficient condition that the two pencils of linear congruences whose directrices are corresponding tangent lines of S and s should cut off the same projectivities on the pencils of lines tangent to S and s at corresponding points is that the y-congruence satisfy (12). But this says that the y-congruence is a W-congruence.†

A necessary and sufficient condition that the plane common to all the tangent S_3's to M along G should lie on Q, that is a necessary and sufficient condition that the tangent linear congruences to m at lines of a pencil g have in common a plane of lines or a bundle of lines, is that the plane of the first three points of (9) lie on Q. But this plane is the osculating plane to the curve of α, along which the lines of M are tangent. Hence the above condition becomes that the curves on α, along which the lines of M are tangent, should have their osculating planes lying on Q.

If α is a ruled surface the two M's defined are first, M_1 consists of the tangent lines to α along the rulings. Then M_1 is a two-dimensional variety, α, and not a three as we have considered. Second, M_2 consists of the other set of lines tangent to α and lying on Q. The corresponding y-congruence in S_3 is made up of ∞^1 pencils of lines, and the surfaces S and s are a curve and a developable, respectively. m_1 is a family of the sort described in Part I. If α is a developable it is a plane.

If α is cut by ∞^1 planes of Q in curves, M_1 will consist of the lines tangent to these plane curves. Conversely, if the lines of M can be grouped into the tangents to ∞^1 plane curves, these planes lie on Q. In the corresponding case in S_3 the y-congruence is made up of the lines of ∞^1 cones or the tangent lines to ∞^1 plane curves. Then either S will be a curve and s a non-specialized surface, or s will be a developable and S a non-specialized surface. Conversely, if one of the surfaces is a curve or a developable, the surface α can be cut by ∞^1 planes of Q in curves. If both S and s are curves, or both are developables, the two sets of lines tangent to α and lying on Q will be tangent to α along plane curves.

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* Segre, "Su una classe di superficie degli iperspazi legata colle equazione lineare alle derivate parziali di 2 ordine," *Atti della R. Accad. delle Scienze di Torino*, Vol. XLIX (1913–14), p. 215.

† Darboux, "Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal," Edition II (1889), p. 345.

If the lines of M, instead of being tangent to a surface, all cut a curve on Q, ∞^1 of them pass through each point of this curve. Then all the pencils of m have a line in common with a ruled surface. Hence, S and s are this surface, but S and s do not coincide in the sense that the center of the pencil of m on S is the point of contact of the pencil with s. The y-congruence also reduces to this surface.

8. It may happen in S_5 that the tangent S_3's to M along G coincide, that is, there is a fixed tangent S_3 to M along every generator. Then M is said to be developable of the second kind. If M is a developable of the second kind, the α and β surfaces of (3) satisfy two linear homogeneous partial differential equations as follows:

$$\left.\begin{aligned} A_1\alpha_i+B_1\beta_i+C_1\alpha_{i_u}+D_1\beta_{i_u}+E_1\alpha_{i_v}+F_1\beta_{i_v}=0,\\ A_2\alpha_i+B_2\beta_i+C_2\alpha_{i_u}+D_2\beta_{i_u}+E_2\alpha_{i_v}+F_2\beta_{i_v}=0,\end{aligned}\right\} \quad i=0,\,1,\,\ldots\ldots,\,5, \qquad (13)$$

where A_j, B_j, $\ldots\ldots$, F_j, $j=1$, 2 are functions of u and v, and not all identically zero. Conversely, if α and β satisfy (13), M is a developable of the second sort. For under these circumstances the eight points of (6) all lie in an S_3, the tangent S_3 to M along G. Accordingly, if M is developable of the second kind, all the S_3's determined by G and the lines of M infinitely near G coincide. In fact, all the lines of M infinitely near G under these hypotheses, lie in the two planes of Q that contain G. For they all lie in the tangent S_3 to M along G which cuts Q in the two planes containing G.

On the other hand, if all the lines of M infinitely near G intersect G, M is a developable of the second kind. Thus, in S_3 if m is developable of the second kind, all the lines of pencils of m infinitely near g, lie in a special linear congruence with intersecting but distinct directrices which belong to the pencil g. Every pencil of m infinitely near g has a line in common with g and, conversely, if every pencil of m infinitely near g has a line in common with g the m-complex is developable of the second kind.

9. Segre has shown* that if M is developable of the second kind, all the lines of M are tangent to two surfaces of Q, or cut a curve of Q and are tangent to a surface, or cut two curves of Q. Since we know from No. 8 that if

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* Segre, "Preliminari ....," No. 29.

m is developable of the second sort all the pencils of m infinitely near g have a line in common with g, we see that either they have their centers in the plane of g or their planes pass through the center of g. In either case S and s coincide. Hence, if m is developable of the second kind, S and s coincide. See (11). Conversely, if S and s coincide, m is developable of the second kind. By the theorem of Segre every pencil of m is tangent to two congruences (or tangent to a congruence, and has a line in common with a ruled surface, or has a line in common with two ruled surfaces). That is there exist two congruences, excluding for the moment the other two cases, such that every pencil of m has two infinitely near lines in common with each. Thus S must be the focal surfaces (coincident) for both of them, and they consist of the tangent lines to S along the two sets of asymptotic lines.

If the lines of M all cut a curve and are tangent to a surface of Q, the tangents to one set of asymptotics of S involve only ∞^1 lines and therefore S is a ruled surface. Similarly, if the lines of M all cut two curves, S has two sets of rulings and is therefore a quadric.

PART III.

10. A three-parameter family of lines on Q in S_5 may generate either a V_3 or a V_4. If they generate a V_3, that V_3 is either the intersection of Q with an S_4, or consists of ∞^1 planes imbedded on Q. We will demonstrate this in two parts. First let us assume that the V_3 is developable of the second kind. That is the tangent S_3 to V_3 along G is fixed. Through every point of V_3 there are ∞^1 lines of V_3, which lie in the tangent S_3 to V_3 at that point. But this S_3 is tangent to Q along G, and therefore cuts Q in the two planes that contain G. Hence the lines of V_3 in that S_3 lie in one or two planes of Q, and V_3 consists of ∞^1 planes of Q.

If V_3 is not developable of the second kind the tangent S_3's along G cut it in ∞^1 cones of lines which are distinct and V_3 is the locus of these cones. Through every point of V_3 there are ∞^1 lines and every line cuts ∞^2 others. Hence some line besides G in one of these cones cuts some other cone, and the two S_3's of these cones have in common besides G another point and therefore a plane. Hence all the tangent S_3's to V_3 along G lie in an S_4 according to No. 5. Hence V_3 is the intersection of Q with this S_4. Hence, in line space a three-parameter family of pencils of lines either involves all the lines of space or else ∞^3 of them. If they all lie in a complex, that complex is either a linear complex or else is made up of the lines in ∞^1 planes or through ∞^1 points.

11. In what follows let M_4 represent the locus of ∞^3 lines of Q which do not lie on a V_3. Then M_4 coincides with Q. Hence all the tangent S_4's to M_4 along a generator G have an S_3 in common. Therefore, according to Segre,* all the lines of M_4 are tangent to two V_3's, or either V_3 may be replaced by a director variety of less dimension.

The S_3's of G and the lines of M_4 infinitely near G are ∞^2 in number and may be grouped into ∞^1 pencils of S_3's whose axial planes themselves form a pencil of planes about G. A proof of this theorem follows M_4 can be represented analytically by means of

$$x_i = \alpha_i(u,\, v,\, w) + t\beta_i(u,\, v,\, w) \qquad i = 0, 1, 2, \ldots, 5, \tag{14}$$

where α and β are the two tangent or director varieties mentioned in the theorem of Segre. Therefore

$$\left.\begin{aligned} \sigma\alpha &= \lambda\beta + \mu\beta_u + \nu\beta_v + \rho\beta_w, \\ \sigma\beta &= \lambda'\alpha + \mu'\alpha_u + \nu'\alpha_v + \rho'\alpha_w. \end{aligned}\right\} \tag{15}$$

The lines of M_4 infinitely near G are the lines from

$$(\alpha + \alpha_u\Delta u + \alpha_v\Delta v + \alpha_w\Delta w) \quad \text{to} \quad (\beta + \beta_u\Delta u + \beta_v\Delta v + \beta_w\Delta w),$$

where Δu, Δv, and Δw are infinitesimal. Hence the S_3's of G and the lines of M_4 infinitely near G are the S_3's of

$$(\alpha), \quad (\beta), \quad (\alpha_u\Delta u + \alpha_v\Delta v + \alpha_w\Delta w), \quad (\beta_u\Delta u + \beta_v\Delta v + \beta_w\Delta w). \tag{16}$$

As we vary the ratios $\Delta u : \Delta v : \Delta w$ in (16) we get the ∞^2 S_3's determined by G and the lines of M_4 infinitely near G. Call the plane of (α_u), (α_v), and (α_w) the plane π_α and similarly the plane π_β is the plane of (β_u), (β_v), and (β_w). According to (15) G cuts π_α in P_α and cuts π_β in P_β. Points represented by the same ratios of Δu, Δv, and Δw are corresponding points in the planes π_α and π_β. A line in π_α corresponds to a line in π_β. Let H be a line in π_α through P_α, and H' the corresponding line in π_β. Let K be a line joining corresponding points of H and H'. Then the S_3 of G and K is one of the S_3's of (16), and it contains the plane of G and H. As K moves along H these S_3's form a pencil of S_3's about the plane GH. But H was any line of π_α through P_α. Hence these planes themselves form a pencil about the line G. Hence the theorem. Similarly the S_3's of G and the lines of M_4 infinitely near G can be grouped into ∞^1 pencils of S_3's whose axial planes are determined by G and lines in π_β through P_β, that is, the axial planes form a pencil of planes about G in the tangent S_3's to π_α and π_β.

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* Segre, "Preliminari ....," No. 29.

In line space the preceding facts give us that: All the tangent linear complexes to m_4 at lines of a pencil g are special, with the lines of contact of the pencil g as axes, and have in common a linear congruence whose directrices belong to the pencil g. The linear congruences determined by g and the pencils of m_4 infinitely near g can be grouped into ∞^1 pencils of congruences having in common besides g a second pencil h which has a line in common with g. The locus of these pairs of pencils is a special linear congruence whose directrices are coincident. This last congruence is tangent to α or β at the line which g has in common with α or β, where α and β are line complexes to which all the pencils of m_4 are tangent (or congruences or ruled surfaces with which every pencil of m_4 has a line in common). The m_4 may be considered as the locus of ∞^1 m's of the first developable kind as in Part II.

The ordering of M_4 into ∞^1 m's of the first developable kind can be accomplished in only two ways corresponding to the two ways of ordering the S_3's of G, and the lines of M_4 infinitely near G into ∞^1 pencils of S_3's. For let $f(u, v, w)=0$ be any relation on the parameters u, v, w which yields a developable V_3 of the first kind on M_4. Then

$$f_u \Delta u + f_v \Delta v + f_w \Delta w = 0, \tag{17}$$

and (17) represents a line H in π_α, and a corresponding line H' in π_β. Lines joining corresponding points of H and H' form a regulus, and the S_3's of G and the lines of this regulus are the S_3's of G and the lines of M_4 infinitely near G, and lying on the V_3, determined by $f=0$. If this V_3 is to be developable of the first kind all these S_3's must have a plane in common and G must cut this regulus. Let us assume that this intersection takes place outside of π_α and π_β. Then the line of the regulus which cuts G cuts π_α in Q_α and π_β in Q_β. Hence P_α, P_β, Q_α, and Q_β are coplanar, and π_α, π_β, and G all lie in the same S_4. Hence the points (α), (β), (α_u), (α_v), (α_w), (β_u), (β_v), (β_w) all lie in an S_4, and the tangent S_4 to M_4, and therefore to Q along G is fixed, which is impossible. Hence the assumption that G cuts the above regulus in a point outside of π_α and π_β is false. Hence H passes through P_α or H' passes through P_β and the two cases of the preceding paragraph are unique.

Hence the only f's which give developable V_3's of the first kind are those for which the line (17) passes through P_α or P_β. The coordinates of these points may be obtained from (15), the point $(\sigma\alpha - \lambda\beta)$ is the point of G in π_β, and therefore has for coordinates $\mu : \nu : \rho$ which is the point P_β, similarly P_α is

$\mu' : \nu' : \rho'$. Hence the developables of the first kind on M_4 and on m_4 are given by one of the following equations:

$$\left.\begin{aligned} \mu f_u + \nu f_v + \rho f_w &= 0, \\ \mu' f_u + \nu' f_v + \rho' f_w &= 0. \end{aligned}\right\} \quad (18)$$

If α or β or both are surfaces instead of V_3's, there will exist a relation on the three parameters of that variety and one of them can be eliminated. Hence either μ, ν or ρ is zero, or one of μ', ν', ρ' is zero, or both. This will not affect the above reasoning. If either α or β is a curve, the pencils of S_3's determined by G and the lines of the regulus from H to H' collapse into single S_3's, for then either H or H' is a point. That is, the M_4 can be grouped into ∞^1 M's of the second developable kind.

Thus M_4's may be classified according to the type of differential equations (15) that they satisfy. If all the coefficients μ, ν, ρ and μ', ν', ρ' are different from zero, M_4 is made up of lines tangent to two V_3's. If one of these coefficients is zero, the lines of M_4 are tangent to a V_3 and cut a surface. If one of the primed and one of the unprimed coefficients are zero, the lines of M_4 cut two surfaces. If two of the primed or two of the unprimed coefficients are zero, the lines of M_4 are tangent to a V_3 and cut a curve. Other cases can not occur.

Similarly m_4's may be grouped according to the type of differential equations (15) the α and β complexes satisfy. If none of these coefficients is zero, the pencils of m_4 are tangent to two complexes. If one is zero they are tangent to a complex and have a line in common with a congruence. If two are zero in different equations they have a line in common with two congruences. If two are zero in the same equation they are tangent to a complex and have a line in common with a ruled surface. In the last case the pencils can be grouped into ∞^1 m's of the second developable type.

12. If the lines of M_4 can be grouped into ∞^1 M's of the second developable type, there must exist an $f(u, v, w) = 0$ for which the tangent S_3 along G is fixed, and therefore, lies in both the tangent S_4 at α, viz., $(\alpha x) = 0$, and the tangent S_4 at β, viz., $(\beta x) = 0$. Hence the last two points of (16) must lie in the S_3 of intersection of $(\alpha x) = 0$ with $(\beta x) = 0$ for all values of $\Delta u : \Delta v : \Delta w$ which satisfy (17). These four conditions reduce to

$$(\alpha\beta_u)\Delta u + (\alpha\beta_v)\Delta v + (\alpha\beta_w)\Delta w = 0 \text{ by virtue of (17).}$$

Hence

$$(\alpha\beta_u) : (\alpha\beta_v) : (\alpha\beta_w) = f_u : f_v : f_w. \quad (19)$$

And (19) is a necessary and sufficient condition that the lines of M_4 be susceptible to an arrangement into ∞^1 M's of the second developable kind. Furthermore (19) gives us the f's that are developable.

Hence (19) is a necessary and sufficient condition that the pencils of m_4 be capable of grouping into ∞^1 m's of the second developable type, that is whose focal surfaces, S and s, coincide, or such that the locus of the centers of the pencils and the envelope of their planes coincide.

The Pfaffian differential equation

$$X(x, y, z)\,dx + Y(x, y, z)\,dy + Z(x, y, z)\,dz = 0 \tag{20}$$

defines precisely a three-parameter family of pencils of lines in ordinary space. (19) is the condition of integrability. The preceding work shows how to pick out in Klein coordinates the two-parameter families of pencils of lines for which the centers of the pencils lie on a surface S, and their planes are tangent to a surface s, where S and s are of such a nature that lines joining corresponding points of each are tangent to both in the non-integrable case. Voss [*] has discussed three-parameter families of pencils from this point of view.

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[*] Voss, "Zur Theorie den allgemeinen Punktebenesysteme," *Math. Annalen*, Vol. XXIII, p. 45.